

On Hamiltonian Bypasses in Digraphs with the Condition of Y. Manoussakis

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Abstract

Let D be a strongly connected directed graph of order $n \geq 4$ vertices which satisfies the following condition for every triple x, y, z of vertices such that x and y are non-adjacent: If there is no arc from x to z , then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x , then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2$. In [15] (J. of Graph Theory, Vol.16, No. 5, 51-59, 1992) Y. Manoussakis proved that D is Hamiltonian. In [9] it was shown that D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n - 1$) or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities of $n/2$ and $n/2$. In this paper we show that D contains also a Hamiltonian bypass, (i.e., a subdigraph obtained from a Hamiltonian cycle by reversing exactly one arc) or D is isomorphic to one tournament of order 5.

Keywords: Digraphs, cycles, Hamiltonian cycles, Hamiltonian bypasses.

1 Introduction

The directed graph (digraph) D is Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle that includes every vertex of D . A Hamiltonian bypass in D is a subdigraph obtained from a Hamiltonian cycle by reversing exactly one arc. We recall the following well-known degree conditions (Theorems 1-6) that guarantee that a digraph is Hamiltonian.

Theorem 1 (Nash-Williams [17]). Let D be a digraph of order n such that for every vertex x , $d^+(x) \geq n/2$ and $d^-(x) \geq n/2$, then D is Hamiltonian.

Theorem 2 (Ghouila-Houri [14]). Let D be a strong digraph of order n . If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.

Theorem 3 (Woodall [19]). Let D be a digraph of order $n \geq 2$. If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y , then D is Hamiltonian.

Theorem 4 (Meyniel [16]). Let D be a strong digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in D , then D is Hamiltonian.

It is easy to see that Meyniel's theorem is a common generalization of Ghouila-Houri's and Woodall's theorems. For a short proof of Theorem 1.3, see [5].

C. Thomassen [18] (for $n = 2k + 1$) and S. Darbinyan [6] (for $n = 2k$) proved the following:

Theorem 5 [18, 6]. If D is a digraph of order $n \geq 5$ with minimum degree at least $n - 1$ and with minimum semi-degree at least $n/2 - 1$, then D is Hamiltonian (unless some extremal cases which are characterized).

In view of the next theorems we need the following definitions.

Definition 1 [15]. Let k be an integer. A digraph D of order $n \geq 3$ satisfies the condition A_k if and only if for every triple of vertices x, y, z such that x and y are non-adjacent: If there is no arc from x to z , then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2 + k$. If there is no arc from z to x , then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2 + k$.

Definition 2. Let D_0 denote any digraph of order $n \geq 5$, n odd, such that $V(D_0) = A \cup B$, where $A \cap B = \emptyset$, A is an independent set with $(n + 1)/2$ vertices, B is a set of $(n - 1)/2$ vertices inducing any arbitrary subdigraph, and $e(A, B) = (n + 1)(n - 1)/2$. D_0 satisfies the condition A_{-1} , but has no Hamiltonian bypass.

Definition 3. For any $k \in [1, n - 2]$ let D_1 denote a digraph of order $n \geq 4$, obtained from K_{n-k}^* and K_{k+1}^* by identifying a vertex of the first with a vertex of the second. D_1 satisfies the condition A_{-1} , but has no Hamiltonian bypass.

Definition 4. By $T(5)$ we denote a tournament of order 5 with vertex set $V(T(5)) = \{x_1, x_2, x_3, x_4, y\}$ and arc set $A(T(5)) = \{x_i x_{i+1} / i \in [1, 3]\} \cup \{x_4 x_1, x_1 y, x_3 y, y x_2, y x_4, x_1 x_3, x_2 x_4\}$. $T(5)$ satisfies condition A_0 , but has no Hamiltonian bypass.

Theorem 6 (Manoussakis [15]). If a strong digraph D satisfies the condition A_0 , then D is Hamiltonian.

In [4] it was proved that if a digraph D satisfies the condition of Nash-Williams' or Ghouila-Houri's or Woodall's theorem, then D contains a Hamiltonian bypass. In [4] also proved the following theorem:

Theorem 7 (Benhocine [4]). Every strongly 2-connected digraph of order n and minimum degree at least $n - 1$ contains a Hamiltonian bypass, unless D is isomorphic to a digraph of type D_0 .

In [7] the following theorem was proved:

Theorem 8 (Darbinyan [7]). Let D be a strong digraph of order $n \geq 3$. If $d(x) + d(y) \geq 2n - 2$ for all pairs of non-adjacent vertices in D , then D contains a Hamiltonian bypass unless it is isomorphic to a digraph of the set $D_0 \cup \{D_1, T_5, C_3\}$, where C_3 is a directed cycle of length 3.

For $n \geq 3$ and $k \in [2, n]$, $D(n, k)$ denotes the digraph of order n obtained from a directed cycle C of length n by reversing exactly $k - 1$ consecutive arcs. In [7, 8] Darbinyan studied the problem of the existence of $D(n, 3)$ in digraphs with condition of Meyniel's theorem and in oriented graphs with large in-degrees and out-degrees.

Theorem 9 (Darbinyan [7]). Let D be a strong digraph of order $n \geq 4$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in D , then D contains a $D(n, 3)$.

Theorem 10 (Darbinyan [8]). Let D be a oriented graph of order $n \geq 10$. If the minimum in-degree and out-degree of D at least $(n - 3)/2$, then D contains a $D(n, 3)$.

In [9] the following theorem was proved:

Theorem 11. Any strongly connected digraph D of order $n \geq 4$ satisfying the condition A_0 contains a pre-Hamiltonian cycle (i.e., a cycle of length $n - 1$) or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities $n/2$ and $n/2$.

In this paper using Theorem 11 we prove the following:

Theorem 12. Any strongly connected digraph D of order $n \geq 4$ satisfying the condition A_0 contains a Hamiltonian bypass unless D is isomorphic to the tournament $T(5)$.

The following two examples show the sharpness of the condition of Theorem 10. The digraph consisting of the disjoint union of two complete digraphs with one common vertex shows that the bound in the above theorem is best possible and the digraph obtained from a complete bipartite digraph after deleting one arc.

2 Terminology and Notations

We shall assume that the reader is familiar with the standard terminology on the directed graphs (digraph) and refer the reader to the monograph of Bang-Jensen and Gutin [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D , we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D . The order of D is the number of its vertices. Often we will write D instead of $A(D)$ and $V(D)$. The arc of a digraph D directed from x to y is denoted by xy . For disjoint subsets A and B of $V(D)$ we define $A(A \rightarrow B)$ as the set $\{xy \in A(D) / x \in A, y \in B\}$ and $A(A, B) = A(A \rightarrow B) \cup A(B \rightarrow A)$. If $x \in V(D)$ and $A = \{x\}$ we write x instead of $\{x\}$. The out-neighborhood of a vertex x is the set $N^+(x) = \{y \in V(D) / xy \in A(D)\}$ and $N^-(x) = \{y \in V(D) / yx \in A(D)\}$ is the in-neighborhood of x . Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A / xy \in A(D)\}$ and $N^-(x, A) = \{y \in A / yx \in A(D)\}$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x . Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of the vertex x in D defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $\langle A \rangle$. The path (respectively, the cycle) consisting of the distinct vertices x_1, x_2, \dots, x_m ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). We say that $x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m) -path. For a cycle $C_k := x_1 x_2 \cdots x_k x_1$ of length k , the subscripts considered modulo k , i.e. $x_i = x_s$ for every s and i such that $i \equiv s \pmod{k}$. A cycle that contains the all vertices of D (respectively, the all vertices of D except one) is a Hamiltonian cycle (respectively, is a pre-Hamiltonian cycle). If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y , $C[x, y]$ denotes the subpath of C from x to y . A digraph D is strongly connected (or, just, strong) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y . For an undirected graph G , we denote by G^* symmetric digraph obtained from G by replacing every edge xy with the pair xy, yx of arcs. $K_{p,q}$ denotes the complete bipartite graph with partite sets of cardinalities p and q . Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). For integers a and b , $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b . By $D(n; 2) = [x_1 x_n; x_1 x_2 \dots, x_n]$ is denoted the Hamiltonian bypass obtained from a Hamiltonian cycle $x_1 x_2 \dots x_n x_1$ by reversing the arc $x_n x_1$.

3 Preliminaries

The following well-known simple Lemmas 1-4 are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proof of our result.

Lemma 1 [13]. Let D be a digraph of order $n \geq 3$ containing a cycle C_m , $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, C_m) \geq m+1$, then D contains a cycle C_k for all $k \in [2, m+1]$.

The following lemma is a slight modification of a lemma by Bondy and Thomassen [5].

Lemma 2. Let D be a digraph of order $n \geq 3$ containing a path $P := x_1x_2 \dots x_m$, $m \in [2, n-1]$ and let x be a vertex not contained in this path. If one of the following conditions holds:

- (i) $d(x, P) \geq m+2$;
- (ii) $d(x, P) \geq m+1$ and $xx_1 \notin D$ or $x_mx_1 \notin D$;
- (iii) $d(x, P) \geq m$, $xx_1 \notin D$ and $x_mx \notin D$,

then there is an $i \in [1, m-1]$ such that $x_ix, xx_{i+1} \in D$ (the arc x_ix_{i+1} is a partner of x), i.e., D contains a path $x_1x_2 \dots x_ixx_{i+1} \dots x_m$ of length m (we say that x can be inserted into P or the path $x_1x_2 \dots x_ixx_{i+1} \dots x_m$ is extended from P with x).

If in Lemma 1 and Lemma 2 instead of the vertex x consider a path Q , then we get the following Lemmas 3 and 4, respectively.

Lemma 3. Let $C_k := x_1x_2 \dots x_kx_1$, $k \geq 2$, be a non-Hamiltonian cycle in a digraph D . Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $D - C_k$. If $d^-(y_1, C_k) + d^+(y_r, C_k) \geq k+1$, then for all $m \in [r+1, k+r]$ the digraph D contains a cycle C_m of length m with vertex set $V(C_m) \subseteq V(C_k) \cup V(Q)$.

Lemma 4. Let $P := x_1x_2 \dots x_k$, $k \geq 2$, be a non-Hamiltonian path in a digraph D . Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $D - P$. If $d^-(y_1, P) + d^+(y_r, P) \geq k + |A(y_1 \rightarrow x_1)| + |A(x_k \rightarrow y_r)|$, then there is a x_i , $i \in [1, k-1]$, such that $x_iy_1, y_rx_{i+1} \in D$ and D contains a path from x_1 to x_k with vertex set $V(P) \cup V(Q)$.

In the proof of Theorem 11 we also need the following lemma which is a simple extension of a lemma by Y. Manoussakis [15].

Lemma 5. Let D be a digraph of order $n \geq 3$ satisfying condition A_0 . Assume that there are two distinct pairs x, y and x, z of non-adjacent vertices in D . If $d(x) + d(y) \leq 2n - a$ for some integer $a \geq 1$, then $d(x) + d(z) \geq 2n - 2 + a/2$. In particular, if $d(x) + d(y) \leq 2n - 2$, then $d(x) + d(z) \geq 2n - 1$.

Definition 5 ([1], [2]). Let $Q = y_1y_2 \dots y_s$ be a path in a digraph D (possibly, $s = 1$) and let $P = x_1x_2 \dots x_t$, $t \geq 2$, be a path in $D - V(Q)$. Q has a partner on P if there is an arc (the partner of Q) x_ix_{i+1} such that $x_iy_1, y_sy_{i+1} \in D$. In this case the path Q can be inserted into P to give a new (x_1, x_t) -path with vertex set $V(P) \cup V(Q)$. The path Q has a collection of partners on P if there are integers $i_1 = 1 < i_2 < \dots < i_m = s+1$ such that, for every $k = 2, 3, \dots, m$ the subpath $Q[y_{i_{k-1}}, y_{i_k-1}]$ has a partner on P .

Lemma 6 ([1], [2], Multi-Insertion Lemma). Let $Q = y_1y_2 \dots y_s$ be a path in a digraph D (possibly, $s = 1$) and let $P = x_1x_2 \dots x_t$, $t \geq 2$, be a path in $D - V(Q)$. If Q has a collection of partners on P , then there is an (x_1, x_t) -path with vertex set $V(P) \cup V(Q)$.

The following lemma is obvious.

Lemma 7. Let D be a digraph of order $n \geq 3$ and let $C := x_1x_2 \dots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D . If a vertex y is not on C and D contains no Hamiltonian bypass, then

- (i) $d^+(y, \{x_i, x_{i+1}\}) \leq 1$ and $d^-(y, \{x_i, x_{i+1}\}) \leq 1$ for all $i \in [1, n-1]$;
- (ii) $d^+(y) \leq (n-1)/2$, $d^-(y) \leq (n-1)/2$ and $d(y) \leq n-1$;

(iii) if $x_k y, y x_{k+1} \in D$, then $x_{i+1} x_i \notin D$ for all $x_i \neq x_k$.

Let D be a digraph of order $n \geq 3$ and let C_{n-1} be a cycle of length $n - 1$ in D . If for the vertex $y \notin C_{n-1}$, $d(y) \geq n$, then we say that C_{n-1} is a good cycle. Notice that, by Lemma 7, if a digraph D contains a good cycle, then D also contains a Hamiltonian bypass.

4 Proof of Theorem 12

In the proof of Theorem 12 we often will use the following definition:

Definition 6. Let $P_0 := x_1 x_2 \dots x_m$, $m \geq 2$, be an (x_1, x_m) -path in D and let the vertices $y_1, y_2, \dots, y_k \in V(D) - V(P_0)$. For $i \in [1, k]$ we denote by P_i an (x_1, x_m) -path in D with vertex set $V(P_{i-1}) \cup \{y_j\}$ (if it exists), i.e, P_i is extended path obtained from P_{i-1} with some vertex y_j , where $y_j \notin V(P_{i-1})$. If $e + 1$ is the maximum possible number of these paths P_0, P_1, \dots, P_e , $e \in [0, k]$, then we say that P_e is extended path obtained from P_0 with vertices y_1, y_2, \dots, y_k as much as possible. Notice that P_i is an (x_1, x_m) -path of length $m + i - 1$ for all $i \in [0, e]$.

Proof of Theorem 12. By Theorem 9 the digraph D contains a cycle of length $n - 1$ or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities of $n/2$ and $n/2$. If D is a complete bipartite digraph then it is easy to see that D has a Hamiltonian bypass. In the sequel, we assume that D contains a cycle of length $n - 1$. Let $C = x_1 x_2 \dots x_{n-1} x_1$ be an arbitrary cycle of length $n - 1$ in D and let $y \notin C$. It is a simple matter to check that for $n = 4$ the theorem is true. Further, let $n \geq 5$. Note that from the condition A_0 and Lemma 5 immediately follows that $d(y) \geq 3$. Now suppose, to the contrary, that D contains no Hamiltonian bypass (by Lemma 7(ii) it is clear that D also contains no good cycle).

For the cycle C and the vertex y we prove the following Claims 1- 7 below.

Claim 1. $d(y, \{x_i\}) \leq 1$ for all $i \in [1, n - 1]$.

Proof. Assume that the claim is not true. Without loss of generality, assume that $d(y, \{x_{n-1}\}) = 2$, i.e., $x_{n-1} y, y x_{n-1} \in D$. By Lemma 7(i), y is not adjacent with x_1 and x_{n-2} . Since $d(y) \geq 3$, we can assume that for some integers $a \geq 1$ and $b \geq 1$ the following hold

$$d(y, \{x_1, x_2, \dots, x_a\}) = d(y, \{x_{n-2}, x_{n-3}, \dots, x_{n-b-1}\}) = 0, \quad (1)$$

and

$$\min\{d(y, \{x_{a+1}\}), d(y, \{x_{n-b-2}\})\} \geq 1 \quad (2)$$

($x_{n-b-2} = x_{a+1}$ is possible). Now from Lemma 7(i) and (1) it follows that

$$d(y) = d(y, \{x_{n-1}\}) + d(y, C[x_{a+1}, x_{n-b-2}]) \leq n - b - a + 1. \quad (3)$$

If there is an (x_{a+1}, x_{n-1}) -path P (respectively, an (x_{n-1}, x_{n-b-2}) -path Q) with vertex set $V(C)$, then, since (2) and $d(y, \{x_{n-1}\}) = 2$, it is easy to see that D contains a Hamiltonian bypass. So we may assume that there is no (x_{a+1}, x_{n-1}) -path and there is no (x_{n-1}, x_{n-b-2}) -path with vertex set $V(C)$. We extend the path $P_0 := C[x_{a+1}, x_{n-1}]$ (respectively, $P_0 := C[x_{n-1}, x_{n-b-2}]$) with vertices x_1, x_2, \dots, x_a (respectively, $x_{n-b-1}, x_{n-b}, \dots, x_{n-2}$) as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_1, x_2, \dots, x_a\}$, $d \in [1, a]$, (respectively, $u_1, u_2, \dots, u_l \in \{x_{n-b-1}, x_{n-b}, \dots, x_{n-2}\}$, $l \in [1, b]$) are not on the extended path P_e . Therefore using Lemma 2(i), we obtain that

$$d(z_i) \leq n + d - 2 \quad \text{and} \quad d(u_j) \leq n + l - 2 \quad (4)$$

for all $i \in [1, d]$ and $j \in [1, l]$. Since $d \leq a + b - 1$ and $l \leq a + b - 1$, from inequalities (3) and (4) it follows that

$$d(y) + d(z_i) \leq 2n - 1 + d - a - b \leq 2n - 2 \quad \text{and} \quad d(y) + d(u_j) \leq 2n - 1 + l - a - b \leq 2n - 2.$$

The last two inequalities contradicts Lemma 5. Claim 1 is proved. \square

Claim 2. $d(y) \leq n - 2$.

Proof. Suppose, on the contrary, that $d(y) \geq n - 1$. Then, by Lemma 7(ii), $d(y) = n - 1$. Using Lemma 7(i) and Claim 1, we obtain that n odd ($n := 2m + 1$), and without loss of generality, we may assume that

$$N^+(y) = \{x_1, x_3, \dots, x_{n-2}\} \quad \text{and} \quad N^-(y) = \{x_2, x_4, \dots, x_{n-1}\}. \quad (5)$$

By Lemma 7(iii),

$$x_{i+1}x_i \notin D \quad \text{for all} \quad i \in [1, n-1]. \quad (6)$$

Case 2.1. There is a x_i such that $d(x_i) \geq n$. Without loss of generality, we may assume that $d(x_1) \geq n$ because of (5). Since D contains no Hamiltonian bypass, it follows that x_1 has no partner on $C[x_3, x_{n-2}]$. From (6), Lemma 2(ii) and

$$n \leq d(x_1) = d(x_1, \{x_2, x_{n-1}, y\}) + d(x_1, C[x_3, x_{n-2}])$$

it follows that $d(x_1, C[x_3, x_{n-2}]) = n - 3$ and $x_1x_3, x_{n-2}x_1 \in D$. If $x_{n-1}x_2 \in D$, then $D(n, 2) = [x_{n-1}x_2; x_{n-1}yx_3x_4 \dots x_{n-2}x_1x_2]$, and if $x_2x_{n-1} \in D$, then $D(n, 2) = [x_2x_{n-1}; x_2yx_1x_3x_4 \dots x_{n-1}]$, which contradicts to our assumption. So, we can assume that x_2, x_{n-1} are non-adjacent. Since $yx_1x_3x_4 \dots x_{n-1}y$ (respectively, $x_{n-2}x_1x_2yx_3 \dots x_{n-2}$) is a cycle of length $n - 1$ which does not contain the vertex x_2 (respectively, x_{n-1}), by Lemma 7(ii), $d(x_2) \leq n - 1$ (respectively, $d(x_{n-1}) \leq n - 1$) and $d^-(x_2) \leq (n - 1)/2 = m$. Now since the triple of vertices x_{n-1}, x_2, y satisfies the condition A_0 , we obtain that

$$3n - 2 \leq d(x_{n-1}) + d(x_2) + d^-(x_2) + d^+(y) \leq 2n - 2 + 2m = 3n - 3,$$

which is a contradiction.

Case 2.2. $d(x_i) \leq n - 1$ for all $i \in [1, n - 1]$. Observe that $d(x_i) + d(x_j) \leq 2n - 2$ for all distinct vertices x_i and x_j . Observe that this together with Lemma 5 implies that every vertex x_i is adjacent with all vertices of D maybe except only one vertex.

Subcase 2.2.1. $x_ix_{i+2} \in D$ for some $i \in [1, n - 1]$. Without loss of generality, assume that $x_1x_3 \in D$. Then

(i) $x_2x_4 \notin D$ (otherwise, if $x_2x_4 \in D$, then $D(n, 2) = [x_2x_3; x_2x_4x_5 \dots x_{n-1}yx_1x_3]$).

(ii) $x_2x_{n-1} \notin D$ (otherwise, if $x_2x_{n-1} \in D$, then $D(n, 2) = [x_2x_{n-1}; x_2yx_1x_3 \dots x_{n-1}]$).

(iii) $x_{n-1}x_2 \notin D$ (otherwise, if $x_{n-1}x_2 \in D$ and $n \geq 6$, then $D(n, 2) = [x_1x_2; x_1x_3x_4yx_5 \dots x_{n-1}x_2]$), and if $x_{n-1}x_2 \in D$ and $n = 5$, then $x_3x_1 \notin D$ and D is isomorphic to $T(5)$).

Therefore, if D is not isomorphic to $T(5)$, then by (ii) and (iii), x_2, x_{n-1} are non-adjacent. Now we will consider the cycle $C' := yx_1x_3x_4 \dots x_{n-1}y$ of length $n - 1$ which does not contain x_2 . By Lemma 7(ii), $d^-(x_2) \leq m$. This together with $d^+(y) = m$, $d(x_2)$ and $d(x_{n-1}) \leq n - 1$ implies that

$$d(x_{n-1}) + d(x_2) + d^-(x_2) + d^+(y) \leq 2n - 2 + 2m = 3n - 3,$$

which contradicts the condition A_0 , since x_2, x_{n-1} are non-adjacent and $yx_2 \notin D$.

Subcase 2.2.2. $x_ix_{i+2} \notin D$ for all $i \in [1, n - 1]$. It is not difficult to see that any x_i cannot be inserted into $C[x_{i+1}, x_{i-1}]$. By Lemma 2(iii), $d(x_i, C[x_{i+2}, x_{i-2}]) \leq n - 5$. Therefore, since $d(x_i, \{y, x_{i-1}, x_{i+1}\}) = 3$, we have that $d(x_i) \leq n - 2$ for all $i \in [1, n - 1]$. By Lemma 5, from this and the above observation we

conclude that D contains no cycle of length two, every vertex x_i is adjacent exactly with $n - 2$ vertices, and hence $d(x_i) = n - 2$ for all x_i .

First we consider the vertex x_2 . Without loss of generality, assume that x_2, x_r are non-adjacent, where $r \in [4, n - 1]$. The triple of vertices x_2, x_r, y satisfies the condition A_0 , since $yx_2 \notin D$. Therefore

$$3n - 2 \leq d(x_r) + d(x_2) + d^-(x_2) + d^+(y) \leq 2n - 4 + (n - 1)/2 + d^-(x_2) \quad (7)$$

and $d^-(x_2) \geq (n + 5)/2 = m + 3$ (recall that $n = 2m + 1$). From this, since x_2 cannot be inserted into $C[x_3, x_1]$ and $x_2x_4 \notin D$, $x_{n-1}x_2 \notin D$, we obtain that

$$N^-(x_2) = \{x_1, x_4, x_5, \dots, x_{r-1}\} \quad \text{and} \quad N^+(x_2) = \{y, x_3, x_{r+1}, x_{r+2}, \dots, x_{n-1}\}. \quad (8)$$

In particular, $r \geq m + 6$ and $x_4x_2 \in D$. Now we consider the vertex x_1 . Without loss of generality, assume that x_1, x_k are non-adjacent, where $k \in [3, n - 2]$. Similarly (7) and (8), we obtain

$$3n - 2 \leq d(x_1) + d(x_k) + d^+(x_1) + d^-(y), \quad d^+(x_1) \geq m + 3,$$

$$N^+(x_1) = \{x_2, x_{n-2}, x_{n-3}, \dots, x_{k+1}\} \quad \text{and} \quad k \leq r - 1.$$

In particular, $x_1x_r \in D$. By symmetry of x_1 and x_3 , we also have that $x_3x_{n-1} \in D$. Now from (5) and (8) we have that $D(n, 2) = [x_3x_{n-1}; x_3x_4 \dots x_{r-1}x_2yx_1x_r \dots x_{n-1}]$. This is contrary to the our assumption and completes the proof of Claim 2. \square

Claim 3. Let $d(y, C[x_{l+1}, x_{k-1}]) = 0$ and y is adjacent with x_l and x_k , where $a + 2 := |C[x_l, x_k]| \geq 3$. Then

(i) if $x_ly, x_ky \in D$ or $yx_l, yx_k \in D$, then there is a vertex $u \in C[x_{l+1}, x_{k-1}]$ such that $d(y) + d(u) \leq 2n - 3$;

(ii) if $x_ly, yx_k \in D$, then there is an (x_k, x_l) -path with vertex set $V(C) - \{u\}$, where u is some vertex of $C[x_{l+1}, x_{k-1}]$ and $d(u) \leq n - 1$. In particular, $d(y) + d(u) \leq 2n - 3$.

(iii) if $x_ly, yx_k \in D$ (or $yx_l, yx_k \in D$ or $x_ly, x_ky \in D$), then there are no x_i and x_j such that $C[x_i, x_j] \neq C[x_l, x_k]$, $b := |C[x_i, x_j]| \geq 3$, $d(y, C[x_{i+1}, x_{j-1}]) = 0$ and a) $x_iy, x_jy \in D$ or b) $yx_i, yx_j \in D$ or c) $x_iy, yx_j \in D$.

Proof. By Claim 1, $d(y) \leq n - a - 1$.

(i). It is not difficult to see that there is no (x_k, x_l) -path with vertex set $V(C)$. We extend the path $P_0 := C[x_k, x_l]$ with vertices $x_{l+1}, x_{l+2}, \dots, x_{k-1}$ as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_{l+1}, x_{l+2}, \dots, x_{k-1}\}$, $d \in [1, a]$, are not on the obtained extended path P_e . Hence using Lemma 2(i) we obtain that $d(z_i) \leq n + d - 2$ (let $u := z_1$). Therefore for all $i \in [1, d]$

$$d(y) + d(z_i) \leq n - a - 1 + n + d - 2 \leq 2n - 3. \quad (9)$$

(ii). Assume, without loss of generality, that $x_{n-1}y, yx_{a+1} \in D$ (i.e., $x_l = x_{n-1}$ and $x_k = x_{a+1}$) and $d(y, C[x_1, x_a]) = 0$ where $a \in [1, n - 4]$. If $a = 1$, then Claim 3(ii) clearly is true. So, we can assume that $a \geq 2$. We extend the path $P_0 := C[x_{a+1}, x_{n-1}]$ with vertices x_1, x_2, \dots, x_a as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_1, x_2, \dots, x_a\}$ are not in the extended path P_e . We claim that $d = 0$ or $d = 1$. Indeed, if $d \geq 2$, then for the vertices z_1 and z_2 inequality (9) holds, which contradicts Lemma 5. Therefore $d = 0$ or $d = 1$. If $d = 1$, then $d(z_1) \leq n - 1$ (let $u := z_1$) and P_e is an (x_{a+1}, x_{n-1}) -path with vertex set $V(C) - \{u\}$, and if $d = 0$, then $e \geq 2$, P_{e-1} is an (x_{a+1}, x_{n-1}) -path with vertex set $V(C) - \{u\}$, where now u is some vertex of $C[x_1, x_a]$, and $d(u) \leq n - 1$ since D contains no good cycle. It is clear that $d(y) + d(u) \leq 2n - 3$.

(iii). Assume that Claim 3(iii) is not true. From Claims 3(i) and 3(ii) it follows that there are two distinct vertices $u \in C[x_{l+1}, x_{k-1}]$ and $v \in C[x_{i+1}, x_{j-1}]$ such that $d(y) + d(u) \leq 2n - 3$ and

$d(y) + d(v) \leq 2n - 3$. These last two inequalities contradicts Lemma 5, since y, u and y, v are two distinct pairs of non-adjacent vertices. Claim 3 is proved. \square

Claim 4. There are no two distinct vertices x_i and x_j such that $x_i y, x_j y \in D$ (or $y x_i, y x_j \in D$), $|C[x_i, x_j]| \geq 3$ and $d(y, C[x_{i+1}, x_{j-1}]) = 0$.

Proof. The proof is by contradiction. Without loss of generality, we may assume that $x_{n-1} y, x_{a+1} y \in D$, $a \geq 1$ and $d(y, C[x_1, x_a]) = 0$. Then $a \in [1, n-4]$ (by Lemma 7(i)) and $y x_{a+2} \in D$ (by Claim 3(iii)). From this it is easy to see that

$$x_i x_{i-1} \notin D \quad \text{for all } i \neq a+2. \quad (10)$$

We will distinguish two cases, according as $a \geq 2$ or $a = 1$.

Case 4.1. $a \geq 2$. Note that $d(y) \leq n - a - 1$ (by Claim 1). We extend the path $P_0 := C[x_{a+1}, x_{n-1}]$ with vertices x_1, x_2, \dots, x_a as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_1, x_2, \dots, x_a\}$, $d \in [1, a]$, are not on the obtained extended path P_e . Using Lemma 2(i), we obtain that $d(z_i) \leq n + d - 2$. Therefore

$$d(y) + d(z_i) \leq 2n - 3 + d - a \leq 2n - 3. \quad (11)$$

This together with Lemma 5 implies that $d = 1$. Let $z_1 := x_k$. Then $d(x_k) \leq n - 1$. First we prove the following Propositions 1 and 2 below

Proposition 1. If $x_i \neq x_k$ with $i \in [1, a]$, then $d(x_i) \geq n + a$ and x_i has a partner on $C[x_{a+2}, x_{n-1}]$ (i.e., x_i can be inserted into $C[x_{a+2}, x_{n-1}]$).

Indeed, the inequality $d(y) + d(x_k) \leq 2n - 2 - a$ (by (11) and $d = 1$) together with Lemma 5 implies that $d(y) + d(x_i) \geq 2n - 1$. Therefore $d(x_i) \geq n + a$, since $d(y) \geq n - a - 1$. It is easy to see that

$$n + a \leq d(x_i) = d(x_i, C[x_{a+2}, x_{n-1}]) + d(x_i, C[x_1, x_{a+1}]) \leq d(x_i, C[x_{a+2}, x_{n-1}]) + 2a.$$

Hence $d(x_i, C[x_{a+2}, x_{n-1}]) \geq n - a \geq |C[x_{a+2}, x_{n-1}]| + 2$, and by Lemma 2(i) the vertex x_i has a partner on $C[x_{a+2}, x_{n-1}]$. \square

Proposition 2. Any two vertices x_i and x_j with $k \leq i < j - 1 \leq a$ (or $1 \leq i < j - 1 \leq k - 1$) are non-adjacent.

Indeed, using Proposition 1 and Multi-Insertion Lemma, we obtain that there is an (x_j, x_i) -Hamiltonian path, say P , and there is an (x_j, x_i) -path, say Q , with vertex set $V(D) - \{x_{j-1}\}$. If $x_j x_i \in D$, then P together with the arc $x_j x_i$ forms a Hamiltonian bypass, and if $x_i x_j \in D$, then Q together with the arc $x_i x_j$ forms a good cycle, since $d(x_{j-1}) \geq n + a$, which contradicts the supposition that D contains no Hamiltonian bypass and good cycle. Therefore x_j and x_i are non-adjacent. \square

Assume first that $k = 1$ (i.e., $x_k = x_1$). From Proposition 2 and (10) it follows that

$$d^-(x_1, C[x_2, x_{a+1}]) = d^+(x_a, C[x_1, x_{a-1}]) = 0. \quad (12)$$

In particular, $x_a x_1 \notin D$. Thus the triple of vertices x_1, y, x_a satisfies condition A_0 . Using (11), $d = 1$, $z_1 = x_1$ and (12), we obtain

$$3n - 2 \leq d(x_1) + d(y) + d^-(x_1) + d^+(x_a) \leq 2n - 2 - a + d^-(x_1) + d^+(x_a),$$

and hence

$$n + a \leq d^-(x_1) + d^+(x_a) = d^-(x_1, C[x_{a+1}, x_{n-1}]) + d^+(x_a, C[x_{a+1}, x_{n-1}]).$$

Now, by Lemma 4, we can insert the path $x_1 x_2 \dots x_a$ into $C[x_{a+1}, x_{n-1}]$ and obtain an (x_{a+1}, x_{n-1}) -path, say R , with vertex set $V(C)$. Therefore, $[x_{a+1} y; Ry]$ is a Hamiltonian bypass, a contradiction.

Assume second that $k \geq 2$ (i.e., $x_k \in C[x_2, x_a]$). From Proposition 2 and (10) it follows that

$$d^-(x_1, C[x_2, x_k]) = 0 \quad \text{and if } k \leq a-1, \quad \text{then } d^+(x_k, C[x_{k+2}, x_{a+1}]) = 0, \quad (13)$$

$$d^-(x_1, C[x_2, x_{a+1}]) \leq a - k + 1 \quad \text{and} \quad d^+(x_k, C[x_1, x_{a+1}]) = 1. \quad (14)$$

In particular, $x_k x_1 \notin D$. The triple of vertices y, x_k, x_1 satisfies the condition A_0 . Hence, using (11), (13) and (14), we obtain

$$3n - 2 \leq d(x_k) + d(y) + d^-(x_1) + d^+(x_k) \leq 2n - 2 - a + d^-(x_1) + d^+(x_k),$$

$$\begin{aligned} n + a \leq d^-(x_1) + d^+(x_k) &= d^-(x_1, C[x_{a+2}, x_{n-1}]) + d^+(x_k, C[x_{a+2}, x_{n-1}]) + \\ &\quad d^-(x_1, C[x_1, x_{a+1}]) + d^+(x_k, C[x_1, x_{a+1}]). \end{aligned}$$

and

$$d^-(x_1, C[x_{a+2}, x_{n-1}]) + d^+(x_k, C[x_{a+2}, x_{n-1}]) \geq n + k - 2 \geq n.$$

Therefore, by Lemma 4, the path $x_1 x_2 \dots x_k$ can be inserted into $C[x_{a+2}, x_{n-1}]$. On the other hand, since every vertex x_i with $i \in [k+1, a]$ has a partner on $C[x_{a+2}, x_{n-1}]$ (Proposition 1) by Multi-Insertion Lemma there exists an (x_{a+2}, x_{n-1}) -path, say R , with vertex set $V(C) - \{x_{a+1}\}$. Therefore, $[x_{a+1}y; x_{a+1}Ry]$ is a Hamiltonian bypass in D , which contradicts the supposition that D has no Hamiltonian bypass.

Case 4.2. $a = 1$. Then x_1 cannot be inserted into $C[x_2, x_{n-1}]$. Therefore by Lemma 2(i), $d(x_1) \leq n - 1$, and hence

$$d(y) + d(x_1) \leq 2n - 3. \quad (15)$$

Recall that $x_2 x_1 \notin D$ and $x_1 x_{n-1} \notin D$ (by (10)). The triples of vertices y, x_1, x_{n-1} and y, x_1, x_2 satisfies condition A_0 . Condition A_0 together with (15) implies that

$$3n - 2 \leq d(x_1) + d(y) + d^+(x_1) + d^-(x_{n-1}) \leq 2n - 3 + d^+(x_1) + d^-(x_{n-1}),$$

and so $d^+(x_1) + d^-(x_{n-1}) \geq n + 1$. A similar argument gives $d^-(x_1) + d^+(x_2) \geq n + 1$.

The last two inequalities and $d(x_1) \leq n - 1$ imply that

$$d^-(x_{n-1}) + d^+(x_2) \geq 2n + 2 - d(x_1) \geq n + 3. \quad (16)$$

From $yx_{n-1} \notin D$ (Claim 1), and (10) we obtain that $d^-(x_{n-1}, \{y, x_1, x_2\}) \leq 1$ and $d^+(x_2, \{y, x_1, x_{n-1}\}) \leq 2$. This together with (16) implies that $n \geq 8$ and

$$d^-(x_{n-1}, C[x_3, x_{n-2}]) + d^+(x_2, C[x_3, x_{n-2}]) \geq n > |C[x_3, x_{n-2}]| + 2.$$

By Lemma 4, we can insert the path $x_{n-1}x_1x_2$ into $C[x_3, x_{n-2}]$ and will obtain an (x_3, x_{n-2}) -path, say P , with vertex set $V(C)$. If $yx_{n-2} \in D$, then $[yx_{n-2}; yP]$ is a Hamiltonian bypass, a contradiction. So, by Lemma 7(i) we can assume that x_{n-2} and y are non-adjacent. From Claim 3(iii) it follows that there exists an integer $b \geq 1$ such that $yx_{n-2-b} \in D$ and $d(y, C[x_{n-b-1}, x_{n-2}]) = 0$. Hence, by Claim 1,

$$d(y) \leq n - 2 - b \quad \text{and} \quad d(y) + d(x_1) \leq 2n - (b + 3). \quad (17)$$

It is clear that $n - b - 2 \neq 4$ (Lemma 7(i)).

Let $n - b - 2 \geq 5$. Then Claim 3(iii) implies that $x_{n-b-3}y \in D$. From (17) and Lemma 5 it follows that for every vertex x_i with $i \in [n - b - 1, n - 2]$ the following inequalities hold

$$d(y) + d(x_i) \geq 2n - 2 + (b + 3)/2 \quad \text{and} \quad d(x_i) \geq n + (3b + 3)/2,$$

and hence, using (10) we obtain

$$d(x_i, C[x_3, x_{n-b-3}]) \geq n + (3b + 3)/2 - (2b + 4) \geq n - (b + 5)/2 \geq |C[x_3, x_{n-b-3}]| + 2 = n - b - 3.$$

Therefore, by Lemma 2(i), every vertex x_i , $i \in [n - b - 1, n - 2]$ has a partner on $C[x_3, x_{n-b-3}]$. By Multi-Insertion Lemma there exists an (x_3, x_{n-b-3}) -path, say R , with vertex set $C[x_3, x_{n-b-3}] \cup C[x_{n-b-1}, x_{n-2}]$. Note that $|R| = n - 5$. From (16) we have

$$\begin{aligned} n + 3 &\leq d^-(x_{n-1}) + d^+(x_2) = d^-(x_{n-1}, R) + d^+(x_2, R) + d^-(x_{n-1}, \{x_1, x_2, y, x_{n-b-2}\}) + \\ &\quad d^+(x_2, \{x_{n-1}, x_1, y, x_{n-b-2}\}), \end{aligned}$$

and, since $d^-(x_{n-1}, \{x_1, x_2, y, x_{n-b-2}\}) \leq 2$ and $d^+(x_2, \{x_{n-1}, x_1, y, x_{n-b-2}\}) \leq 3$,

$$d^-(x_{n-1}, R) + d^+(x_2, R) \geq n - 2 \geq |R| + 2.$$

By Lemma 4 this means that we can insert the path $x_{n-1}x_1x_2$ into R . Therefore there is an (x_3, x_{n-b-3}) -path, say Q , with vertex set $V(C) - \{x_{n-b-2}\}$ and hence, $[yx_{n-b-2}; yRx_{n-b-2}]$ is a Hamiltonian bypass, a contradiction.

Let finally $n - b - 2 = 3$. Then $d(y) = 3$, $d(x_1) \leq n - 1$, $d^-(x_1) \leq n - 3$ and $d^+(x_2) \leq n - 2$. Therefore, since $x_2x_1 \notin D$, by condition A_0 we obtain that

$$3n - 2 \leq d(y) + d(x_1) + d^-(x_1) + d^+(x_2) \leq 3n - 3,$$

which is a contradiction, and completes the proof of Claim 4. \square

Claim 5. Let $x_r y, yx_k \in D$ and $d(y, C[x_{r+1}, x_{k-1}]) = 0$ for some $r, k \in [1, n - 1]$, where $3 \leq |C[x_r, x_k]| \leq n - 2$. Then the vertices y and x_{k+1} are non-adjacent.

Proof. Assume, without loss of generality, that $x_{n-1}y, yx_{a+1} \in D$ (i.e., $x_r = x_{n-1}$ and $x_k = x_{a+1}$) and $d(y, C[x_1, x_a]) = 0$ where $a \in [1, n - 4]$.

Suppose that Claim 5 is not true, i.e., the vertices y and x_{a+2} are adjacent. From Lemma 7(i) it follows that $x_{a+2}y \in D$ and $a + 2 \leq n - 3$. Together with Claim 3(iii) this implies that $yx_{a+3} \in D$. It is easy to see that

$$x_i x_{i-1} \notin D \quad \text{for all } i \neq a + 3. \quad (18)$$

By Claim 3(iii) there exists a vertex $x_j \in C[x_1, x_a]$ such that $d(x_j) \leq n - 1$. Therefore

$$d(y) + d(x_j) \leq 2n - (a + 2). \quad (19)$$

Proposition 3. Let $x_l \neq x_j$ with $i \in [1, a]$ (if $a \geq 2$) be an arbitrary vertex. Then x_l has a partner on $C[x_{a+3}, x_{n-1}]$ and $d(x_l) \geq n + 3a/2$.

Indeed, by Lemma 5 and (19) the following hold

$$d(y) + d(x_l) \geq 2n - 2 + (a + 2)/2 \quad \text{and} \quad d(x_l) \geq n + 3a/2.$$

Hence, since $x_{l+1}x_l \notin D$ (by (18)), we have that

$$n + 3a/2 \leq d(x_l) = d(x_l, C[x_{a+3}, x_{n-1}]) + d(x_l, C[x_1, x_{a+2}]) \leq d(x_l, C[x_{a+3}, x_{n-1}]) + 2a + 1.$$

Therefore

$$d(x_l, C[x_{a+3}, x_{n-1}]) \geq n - a/2 - 1 \geq |C[x_{a+3}, x_{n-1}]| + 2 = n - a - 1,$$

and by Lemma 2(i), x_l has a partner on $C[x_{a+3}, x_{n-1}]$. \square

Now using Proposition 3, (18) and Multi-Insertion Lemma it is not difficult to show that

$$d^+(x_{a+1}, C[x_j, x_a]) = d^-(x_j, C[x_1, x_{j-2}] \cup C[x_{j+1}, x_{a+1}]) = 0,$$

(here if $x_j = x_1$ or x_2 , then $C[x_1, x_{j-2}] = \emptyset$) for otherwise by (18) $a \geq 2$ and D contains a Hamiltonian bypass or a good cycle. In particular, these equalities imply that

$$d^-(x_j, C[x_1, x_{a+2}]) \leq 2 \quad \text{and} \quad d^+(x_{a+1}, C[x_1, x_{a+2}]) \leq j. \quad (20)$$

Note that the triple of vertices y, x_j, x_{a+1} satisfies the condition A_0 , since $x_{a+1}x_j \notin D$ and the vertices y, x_j are non-adjacent. The condition A_0 together with (19) and (20) implies that

$$\begin{aligned} 3n - 2 &\leq d(y) + d(x_j) + d^-(x_j) + d^+(x_{a+1}); \\ n + a &\leq d^-(x_j) + d^+(x_{a+1}) = d^-(x_j, C[x_1, x_{a+2}]) + d^+(x_{a+1}, C[x_1, x_{a+2}]) + \\ &\quad d^-(x_j, C[x_{a+3}, x_{n-1}]) + d^+(x_{a+1}, C[x_{a+3}, x_{n-1}]). \end{aligned}$$

From this and (20) we obtain that

$$d^-(x_j, C[x_{a+3}, x_{n-1}]) + d^+(x_{a+1}, C[x_{a+3}, x_{n-1}]) \geq n + a - 2 - j \geq |C[x_{a+3}, x_{n-1}]| + 2.$$

Therefore, by Lemma 4, the path $x_jx_{j+1} \dots x_{a+1}$ has a partner on $C[x_{a+3}, x_{n-1}]$. This together with Proposition 3 implies that the path $x_1x_2 \dots x_{a+1}$ has a collection of partners on $C[x_{a+3}, x_{n-1}]$, and by Multi-Insertion Lemma there is an (x_{a+3}, x_{n-1}) -path, say R , so that $V(R) = V(C) - \{x_{a+2}\}$. This means that $[x_{a+2}y; x_{a+2}Ry]$ is a Hamiltonian bypass, a contradiction. Claim 5 is proved.

Claim 6. If $x_ly \in D$ and $d(y, C[x_{l+1}, x_{l+a}]) = 0$, where $a \in [1, n-4]$, then $yx_{l+a+1} \notin D$.

Proof. The proof is by contradiction. Without loss of generality, assume that $x_{n-1}y \in D$, $d(y, C[x_1, x_a]) = 0$ and $yx_{a+1} \in D$, where $a \in [1, n-4]$. By Claim 5, the vertices y and x_{a+2} are non-adjacent. If we consider the converse digraph of D we obtain that the vertices x_{n-2} and y also are non-adjacent. It follows from Claim 3(iii) that there is an integer $b \geq 1$ such that $d(y, C[x_{a+2}, x_{a+b+1}]) = 0$ and $x_{a+b+2}y \in D$. Using the fact that $d(y) \geq 3$, Lemma 7(i) and again Claim 3(iii) we obtain that $a + b + 3 \leq n - 3$, $yx_{a+b+3} \in D$, and hence

$$x_i x_{i-1} \notin D \quad \text{for all } i \neq a + b + 3. \quad (21)$$

Notice that (by Claim 1)

$$d(y) \leq n - 2 - a - b. \quad (22)$$

On the other hand, by Claim 3(ii) there is a vertex x_k with $k \in [1, a]$ such that $d(x_k) \leq n - 1$. This together with (22) implies that

$$d(y) + d(x_k) \leq 2n - (a + b + 3). \quad (23)$$

Therefore by Lemma 5, (22) and (23) for every vertex $u \in C[x_1, x_a] \cup C[x_{a+2}, x_{a+b+1}] - \{x_k\}$ the following hold

$$\begin{aligned} d(u) + d(y) &\geq 2n - 2 + (a + b + 3)/2; \\ d(u) &\geq 2n - 2 + (a + b + 3)/2 - n + 2 + a + b = n + 3(a + b + 1)/2; \end{aligned} \quad (24)$$

and, since (21),

$$\begin{aligned} d(u) &= d(u, C[x_1, x_{a+b+2}]) + d(u, C[x_{a+b+3}, x_{n-1}]) \leq d(u, C[x_{a+b+3}, x_{n-1}]) + 2(a + b + 1) - 1; \\ d(u, C[x_{a+b+3}, x_{n-1}]) &\geq n + 1 - (a + b + 1)/2 \geq |C[x_{a+b+3}, x_{n-1}]| + 2 = n - a - b - 1. \end{aligned}$$

Therefore by Lemma 2(i) the vertex u has a partner on $C[x_{a+b+3}, x_{n-1}]$. On the other hand, using this, (24) and Multi-Insertion Lemma it is not difficult to show that

$$d^-(x_k, C[x_1, x_{a+1}]) \leq 1 \quad \text{and} \quad d^+(x_{a+1}, C[x_k, x_{a+b+2}]) = 1.$$

Hence

$$d^-(x_k, C[x_1, x_{a+b+2}]) \leq b+2 \quad \text{and} \quad d^+(x_{a+1}, C[x_1, x_{a+b+2}]) \leq k. \quad (25)$$

Since the triple of vertices y, x_k, x_{a+1} satisfies the condition A_0 , from (23) and (25) it follows that

$$\begin{aligned} 3n-2 &\leq d(y) + d(x_k) + d^-(x_k) + d^+(x_{a+1}) \leq 2n - (a+b+3) + d^-(x_k, C[x_{a+b+3}, x_{n-1}]) + \\ &\quad d^+(x_{a+1}, C[x_{a+b+3}, x_{n-1}]) + b+2+k, \end{aligned}$$

and since $k \leq a$,

$$\begin{aligned} d^-(x_k, C[x_{a+b+3}, x_{n-1}]) + d^+(x_{a+1}, C[x_{a+b+3}, x_{n-1}]) &\geq 3n-2-2n+(a+b+3)-b-2-k = \\ &= n-1+a-k \geq n-1 \geq |C[x_{a+b+3}, x_{n-1}]| + 2. \end{aligned}$$

Therefore by Lemma 4 the path $x_k x_{k+1} \dots x_{a+1}$ has a partner on $C[x_{a+b+3}, x_{n-1}]$. Thus we have shown that the path $x_1 x_2 \dots x_{a+b+1}$ has a collection of partners on $C[x_{a+b+3}, x_{n-1}]$. From Multi-Insertion Lemma it follows that there exists an (x_{a+b+3}, x_{n-1}) -path, say R , with vertex set $V(C) - \{x_{a+b+2}\}$. Hence, $[x_{a+b+2}y; x_{a+b+2}Ry]$ is a Hamiltonian bypass, which is a contradiction and completes the proof of Claim 6. \square

Claim 7. If $yx_l \in D$ and $d(y, C[x_{l+1}, x_{l+a}]) = 0$ with $a \in [1, n-4]$, then $x_{l+a+1}y \notin D$.

Proof. Suppose that the claim is not true. Without loss of generality, assume that $yx_{n-1} \in D$, $d(y, C[x_1, x_a]) = 0$ and $x_{a+1}y \in D$, where $a \in [1, n-4]$. Notice that $d(y) \leq n-a-1$ by Claim 1. Lemma 7(i) and Claims 4 and 6 imply that $yx_{a+2} \in D$ and $x_{n-2}y \in D$. From this it is easy to see that $x_i x_{i-1} \notin D$ for all $i \in [1, n-1]$.

First we prove the following.

Proposition 4. If $d(x_j) \geq n+a-1$ with $x_j \in C[x_1, x_a]$, then x_j has a partner on $C[x_{a+2}, x_{n-1}]$ and on $C[x_{a+1}, x_{n-2}]$.

Proof of Proposition 4. Since $x_{j+1}x_j \notin D$, it follows that $d(x_j, C[x_1, x_{a+1}]) \leq 2a-1$. Therefore from

$$n+a-1 \leq d(x_j) = d(x_j, C[x_1, x_{a+1}]) + d(x_j, C[x_{a+2}, x_{n-1}])$$

we obtain that

$$d(x_j, C[x_{a+2}, x_{n-1}]) \geq n-a \geq |C[x_{a+2}, x_{n-1}]| + 2 = n-a,$$

and hence, by Lemma 2(i) x_j has a partner on $C[x_{a+2}, x_{n-1}]$. A similar discussion holds for the path $C[x_{a+1}, x_{n-2}]$ and so the proposition is proved. \square

Now we will consider the following cases.

Case 7.1. $a \geq 2$ and $d(x_k) \leq n+a-1$ for some $x_k \in C[x_1, x_a]$. Then, since $d(y) \leq n-a-1$,

$$d(y) + d(x_k) \leq 2n-2. \quad (26)$$

Let $x_j \neq x_k$ with $j \in [1, a]$ be an arbitrary vertex. From Lemma 5 and (26) it follows that

$$d(y) + d(x_j) \geq 2n-1 \quad \text{and} \quad d(x_j) \geq n+a. \quad (27)$$

Without loss of generality, we can assume that $k \geq 2$ (otherwise we consider the converse digraph of D). From Proposition 4 it follows that

$$d^-(x_{n-1}, C[x_1, x_k]) = 0 \quad \text{and} \quad d^+(x_k, C[x_{n-1}, x_a]) \leq 1 \quad (28)$$

(for otherwise, using Multi-Insertion Lemma, we obtain that D contains a Hamiltonian bypass or a good cycle). The triple of vertices y, x_k, x_{n-1} satisfies the condition A_0 , since y, x_k are non-adjacent and $x_k x_{n-1} \notin D$. Therefore using (26) and (28) we obtain that

$$3n - 2 \leq d(y) + d(x_k) + d^-(x_{n-1}) + d^+(x_k) \leq 2n - 2 + 1 + (a - k + 1) +$$

$$d^-(x_{n-1}, C[x_{a+1}, x_{n-2}]) + d^+(x_k, C[x_{a+1}, x_{n-2}])$$

and

$$d^-(x_{n-1}, C[x_{a+1}, x_{n-2}]) + d^+(x_k, C[x_{a+1}, x_{n-2}]) \geq n - 2 - a + k \geq |C[x_{a+1}, x_{n-2}]| + 2 = n - a.$$

Hence by Lemma 4 we have that the path $x_{n-1}x_1x_2 \dots x_k$ has a partner on $C[x_{a+1}, x_{n-2}]$. This together with (27) and Proposition 4 implies that the path $x_{n-1}x_1x_2 \dots x_a$ has a collection of partners on $C[x_{a+1}, x_{n-2}]$, and hence by Multi-Insertion Lemma there is a (x_{a+1}, x_{n-2}) -path, say R , with vertex set $V(C)$. Therefore $[x_{a+1}y; Ry]$ is a Hamiltonian bypass, a contradiction.

Case 7.2. $a \geq 2$ and $d(x_j) \geq n + a$ for all $x_j \in C[x_1, x_a]$. By Proposition 4 every vertex x_j with $j \in [1, a]$ has a partner on $C[x_{a+2}, x_{n-1}]$ and on $C[x_{a+1}, x_{n-2}]$. Therefore by Multi-Insertion Lemma, x_{n-1} (respectively, x_{a+1}) has no partner on $C[x_{a+1}, x_{n-2}]$ (respectively, on $C[x_{a+2}, x_{n-1}]$) because of $x_{a+1}y$ and $x_{n-2}y \in D$ (respectively, yx_{a+2} and $yx_{n-1} \in D$). By Lemma 2(i) this means that

$$d(x_{n-1}, C[x_{a+1}, x_{n-2}]) \leq n - a - 1 \quad \text{and} \quad d(x_{a+1}, C[x_{a+2}, x_{n-1}]) \leq n - a - 1. \quad (29)$$

On the other hand, using Proposition 4 and Multi-Insertion Lemma, one can show that x_{n-1}, x_{a+1} are non-adjacent and

$$d^-(x_1, C[x_2, x_{a+1}]) = d^+(x_{a+1}, C[x_1, x_a]) = 0, \quad (30)$$

$$d(x_{n-1}, \{x_1, x_2, \dots, x_a, y\}) = d(x_{a+1}, \{x_1, x_2, \dots, x_a, y\}) = 2,$$

since D contains no Hamiltonian bypass and good cycle. The last two equalities together with (29) gives

$$d(x_{n-1}) \leq n - a + 1 \quad \text{and} \quad d(x_{a+1}) \leq n - a + 1. \quad (31)$$

Now using the condition A_0 , (30) and (31) we obtain that

$$3n - 2 \leq d(x_{n-1}) + d(x_{a+1}) + d^-(x_1) + d^+(x_{a+1}) \leq 2n - 2a + 2 + d^-(x_1) + d^+(x_{a+1})$$

and

$$n + 2a - 4 \leq d^-(x_1) + d^+(x_{a+1}) = d^-(x_1, C[x_{a+2}, x_{n-1}]) + d^+(x_{a+1}, C[x_{a+2}, x_{n-1}]) +$$

$$d^-(x_1, C[x_2, x_{a+1}]) + d^+(x_{a+1}, C[x_1, x_a] \cup \{y\}),$$

$$d^-(x_1, C[x_{a+2}, x_{n-1}]) + d^+(x_{a+1}, C[x_{a+2}, x_{n-1}]) \geq n + 2a - 5 \geq |C[x_{a+2}, x_{n-1}]| + 2 = n - a,$$

since $a \geq 2$. By Lemma 4 the path $x_1x_2 \dots x_{a+1}$ has a partner on $C[x_{a+2}, x_{n-1}]$. Therefore there is an (x_{a+2}, x_{n-1}) -path, say R , with vertex set $V(C)$. So we have that $[yx_{n-1}; yR]$ is a Hamiltonian bypass, which is contradiction. This contradiction completes the discussion of the case $a \geq 2$.

Case 7.3. $a = 1$. It is easy to see that the arc x_1x_2 has no partner on $C[x_3, x_{n-1}]$. Applying Lemma 4 to the arc x_1x_2 and to the path $C[x_3, x_{n-1}]$ we obtain that

$$d^-(x_1) + d^+(x_2) = d^-(x_1, C[x_3, x_{n-1}]) + d^+(x_2, C[x_3, x_{n-1}]) + d^-(x_1, \{y, x_2\}) +$$

$$d^+(x_2, \{y, x_1\}) \leq n - 1, \quad (32)$$

since $d^-(x_1, \{y, x_2\}) = 0$ and $d^+(x_2, \{y, x_1\}) = 1$. Note that the triple of vertices y, x_1, x_2 satisfies condition A_0 since x_1, y are non-adjacent and $x_2 x_1 \notin D$. This together with $d(y) \leq n - 2$ and (32) implies that

$$3n - 2 \leq d(y) + d(x_1) + d^-(x_1) + d^+(x_2) \leq d(x_1) + 2n - 3$$

and $d(x_1) \geq n + 1$. Now by Proposition 4, x_1 has a partner on $C[x_3, x_{n-1}]$ and $C[x_2, x_{n-2}]$. Therefore by Multi-Insertion Lemma x_2 (respectively, x_{n-1}) has no partner on $C[x_3, x_{n-1}]$ (respectively, $C[x_2, x_{n-2}]$). This means that (by Lemma 2(i))

$$d(x_2) = d(x_2, C[x_3, x_{n-2}]) + d(x_2, \{x_{n-1}, x_1, y\}) \leq n - 1$$

and

$$d(x_{n-1}) = d(x_{n-1}, C[x_3, x_{n-2}]) + d(x_{n-1}, \{x_1, x_2, y\}) \leq n - 1,$$

since x_{n-1}, x_2 are non-adjacent, $x_1 x_{n-1} \notin D$ and $x_2 x_1 \notin D$. Now using condition A_0 , (32) and the last two inequalities we obtain

$$3n - 2 \leq d(x_{n-1}) + d(x_2) + d^-(x_1) + d^+(x_2) \leq 3n - 3,$$

which is a contradiction. Claim 7 is proved. \square

We are now ready to complete the proof of Theorem 12.

From Claims 1 and 2 it follows that there are two distinct vertices x_k, x_l such that $|C[x_k, x_l]| \geq 3$, y is adjacent with x_k, x_l and $d(y, C[x_{k+1}, x_{l-1}]) = 0$. Therefore one of the following cases holds: (i) $x_k y, x_l y \in D$; (ii) $y x_k, y x_l \in D$; (iii) $x_k y, y x_l \in D$; (iv) $y x_k, x_l y \in D$. On the other hand, if D has no Hamiltonian bypass, then Claims 4-7 imply that each of these cases is impossible. Thus we have a contradiction. The proof of Theorem 12 is complete. \square

5 Concluding remarks

Each of Theorems 1-6 imposes a degree condition on all pairs of non-adjacent vertices (or on all vertices). In the following three theorems imposes a degree condition only for some pairs of non-adjacent vertices. In each of the condition (Theorems 13-16) below D is a strongly connected digraph of order n .

Theorem 13 [2] (Bang-Jensen, Gutin, H.Li). Suppose that $\min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for any pair of non-adjacent vertices x, y with a common in-neighbour, then D is Hamiltonian.

Theorem 14 [2] (Bang-Jensen, Gutin, H.Li). Suppose that $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.

Theorem 15 [3] (Bang-Jensen, Guo, Yeo). Suppose that $d(x) + d(y) \geq 2n - 1$ and $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n - 1$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.

Note that Theorem 15 generalizes Theorem 14.

In [10] the following results were proved:

- (i) if the minimum semi-degree of D at least two and D satisfies the condition of Theorem 13 or
- (ii) D is not directed cycle and satisfies the condition of Theorem 14, then either D contains a pre-Hamiltonian cycle or n is even and D is isomorphic to the complete bipartite digraph or to the complete bipartite digraph minus one arc with partite sets of cardinalities $n/2$ and $n/2$.

In [11] proved that if D is not directed cycle and satisfies the condition of Theorem 15, then D contains a pre-Hamiltonian cycle or a cycle of length $n - 2$.

We pose the following problem:

Problem. Characterize those digraphs which satisfy the condition of Theorem 13 (or 14 or 15) but has no Hamiltonian bypass.

In [12] the following theorem was proved:

Theorem 16. Suppose that $\min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for any pair of non-adjacent vertices x, y with a common in-neighbour. If $n \geq 6$ and the minimum out-degree of D at least two and the minimum in-degree of D at least three, then D contains a Hamiltonian bypass.

We believe that Theorem 16 also is true if we require that minimum in-degree at least two instead of three.

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